

# A Result on Best Proximity Point

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**ABSTRACT :** S.Karpagam and Sushama Aggarwal [3] defined p-cyclic contraction to be a map  $T: \bigcup_{i=1}^p A_i \rightarrow \bigcup_{i=1}^p A_i$  satisfying  $T(A_i) \subset T(A_{i+1})$ ,  $1 \leq i \leq p$  where  $A_{p+1} = A_1$  such that for some  $k \in (0,1)$ ,  $d(Tx, Ty) \leq kd(x,y) + (1-k)dist(A_i, A_{i+1})$ ,  $x \in A_i, y \in A_{i+1}$  and obtained best proximity point in one of the sets. Their assumption forces the p distances  $dist(A_i, A_{i+1})$ ,  $1 \leq i \leq p$ , to be same. In this paper we have relaxed the second condition in the definition of p-cyclic contraction in a way which does not force the distances to be same and obtained the best proximity point in one of the sets.

**KEYWORDS :** p-cyclic contraction , best proximity point , uniform convex banach space

## I. INTRODUCTION

The classical and well-known Banach's contraction principle states that every contraction on a complete metric space has a unique fixed point that is realizable as the limit of Picard iterates. Numerous interesting extensions and variants of the aforesaid result exist in the literature. However, the mappings involved in all these results are self-mappings. Fixed point theory is an indispensable tool for solving the equation  $Tx = x$  for a mapping  $T$  defined on a subset of a metric space, a normed linear space or a topological vector space. As a non-self mapping  $T : A \rightarrow B$  does not necessarily have a fixed point, one often tries to determine an element  $x$  which is in some sense closest to  $Tx$ . Best approximation theorems and best proximity point theorems are pertinent in this perspective. A number of authors have studied the conditions which ensure the existence of best proximity point. See for example [1-7].

## II. RELATED WORK

Kirk , Srinivasan and Veeramani [1] proved

**Theorem 1.1:** Let  $\{A_i\}_{i=1,2 \dots p}$  be nonempty closed subsets of a complete metric space and

$T: \bigcup_{i=1}^p A_i \rightarrow \bigcup_{i=1}^p A_i$  be a map such that

1)  $T(A_i) \subset T(A_{i+1})$ ,  $1 \leq i \leq p$  where  $A_{p+1} = A_1$

2) For some  $k \in (0,1)$ ,  $d(Tx, Ty) \leq k(x,y)$ ,  $x \in A_i, y \in A_{i+1}$  then there exists a unique fixed point of  $T$  Eldered and Veeramani [2], weakened the contraction condition for two set in the setting of uniformly convex banach space and obtained the following result on best proximity point:

**Theorem 1.2:** Let  $A$  and  $B$  be nonempty, closed and convex subsets of a uniformly convex banach space.

$T: A \cup B \rightarrow A \cup B$  be such that

1)  $T(A) \subset B$  and  $T(B) \subset A$

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2) for some  $k \in (0,1)$ ,  $\|Tx - Ty\| \leq k\|x - y\| + (1-k) \text{dist}(A,B)$ ,  $x \in A$  and  $y \in B$  then there exists a unique best proximity point  $x \in A$  i.e. a point  $x \in A$  such that  $d(x, Tx) = \text{dist}(A,B)$

S.Karpagam and Sushama Aggarwal [3] defined p-cyclic contraction as a map

$$T: \bigcup_{i=1}^p A_i \rightarrow \bigcup_{i=1}^p A_i \text{ satisfying}$$

1)  $T(A_i) \subset T(A_{i+1})$ ,  $1 \leq i \leq p$  where  $A_{p+1} = A_1$

2) For some  $k \in (0,1)$ ,  $d(Tx, Ty) \leq kd(x,y) + (1-k)\text{dist}(A_i, A_{i+1})$ ,  $x \in A_i, y \in A_{i+1}$  and proved the following theorem :

**Theorem 1.3:** If  $A_1, A_2, \dots, A_p$  are nonempty, closed, convex subsets of a uniformly convex Banach space and  $T: \bigcup_{i=1}^p A_i$

$\rightarrow \bigcup_{i=1}^p A_i$  be a p-cyclic contraction map then there exists an  $x \in A_i$  such that  $d(x, Tx) = \text{dist}(A_i, A_{i+1})$ . Moreover  $T^j x$

is a best proximity point in  $A_{i+j}$ .

Here the assumption forces the p distances,  $\text{dist}(A_i, A_{i+1})$ ,  $1 \leq i \leq p$ , to be same.

In this paper we have relaxed the condition 2) in the definition of p-cyclic contraction in a way which does not the distances to be same and got the best proximity point in one of the sets.

### III. PRELIMINARIES

We will make use of the following lemmas proved in [1]:

**Lemma 2.1:** Let A be a nonempty closed and convex subset, and B be a nonempty closed subset of a uniformly convex Banach space. Let  $\{x_n\}$  and  $\{z_n\}$  be sequences in A and  $\{y_n\}$  be a sequence in B satisfying :

1)  $\|z_n - y_n\| \rightarrow \text{dist}(A,B)$

2) for every  $\epsilon > 0$  there exists a natural number N such that for all  $m > n \geq N$ ,

$$\|x_m - y_n\| \leq \text{dist}(A,B) + \epsilon$$

Then for every  $\epsilon > 0$ , there exists  $N_0$ , such that for all  $m > n \geq N_0$

$$\|x_m - z_n\| \leq \epsilon$$

**Lemma 2.2:** let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space, let  $\{x_n\}$  and  $\{z_n\}$  be sequences in A and  $\{y_n\}$  be a sequence in B satisfying :

1)  $\|x_n - y_n\| \rightarrow \text{dist}(A,B)$

2)  $\|z_n - y_n\| \rightarrow \text{dist}(A,B)$

then  $\|x_n - z_n\| \rightarrow 0$

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## IV. MAIN RESULTS

**Theorem 3.1:** Let  $A_1, A_2, \dots, A_p$  be nonempty, closed, convex subsets of a uniformly convex Banach space and  $T: \bigcup_{i=1}^p A_i$

$\rightarrow \bigcup_{i=1}^p A_i$  be a map satisfying

1)  $T(A_i) \subset T(A_{i+1})$ ,  $1 \leq i \leq p$  where  $A_{p+1} = A_1$

2) For some  $k \in (0, 1)$ ,  $d(Tx, Ty) \leq kd(x, y) + (1-k) \max\{\text{dist}(A_i, A_{i+1}) : 1 \leq i \leq p\}$ ,  $x \in A_i, y \in A_{i+1}$  then there exists  $x \in A_j$  such that  $d(x, Tx) = \text{dist}(A_j, A_{j+1})$

where  $\text{dist}(A_j, A_{j+1}) = \max\{\text{dist}(A_i, A_{i+1}) : 1 \leq i \leq p\}$

To prove the theorem we first prove the following lemmas:

**Lemma 3.2:** Let  $A_1, A_2, \dots, A_p$  be nonempty, closed subsets of a metric space and  $T: \bigcup_{i=1}^p A_i \rightarrow \bigcup_{i=1}^p A_i$  be a map

satisfying the conditions of the theorem 3.1, then for  $x, y \in A_j$ ,  $d(T^{pn} x, T^{pn+1} y) \rightarrow \text{dist}(A_j, A_{j+1})$

**proof:**  $\text{dist}(A_j, A_{j+1}) \leq d(T^{pn} x, T^{pn+1} y) \leq kd(T^{pn-1} x, T^{pn} y) + (1-k) \text{dist}(A_j, A_{j+1})$

$$\leq k[kd(T^{pn-2} x, T^{pn-1} y) + (1-k) \text{dist}(A_j, A_{j+1})] + (1-k) \text{dist}(A_j, A_{j+1})$$

$$= k^2 d(T^{pn-2} x, T^{pn-1} y) + (1-k^2) \text{dist}(A_j, A_{j+1})$$

$\leq \dots$

$$k^{pn} d(x, Ty) + (1-k^{pn}) \text{dist}(A_j, A_{j+1}) \rightarrow \text{dist}(A_j, A_{j+1})$$

thus  $d(T^{pn} x, T^{pn+1} y) \rightarrow \text{dist}(A_j, A_{j+1})$

Similarly we can prove that  $d(T^{pn \pm p} x, T^{pn+1} y) \rightarrow \text{dist}(A_j, A_{j+1})$  and  $d(T^{pn+p+1} x, T^{pn} y) \rightarrow \text{dist}(A_j, A_{j+1})$

**Remark 3.3:** In view of lemma 2.2 and lemma 3.2 we have that if  $X$  is uniformly convex Banach space and each  $A_i$  is convex then for  $x \in A_j$ ,

$\|T^{pn} x - T^{pn+1} x\| \rightarrow \text{dist}(A_j, A_{j+1})$  and  $\|T^{pn \pm p} x - T^{pn+1} x\| \rightarrow \text{dist}(A_j, A_{j+1})$  and therefore  $\|T^{pn} x - T^{pn \pm p} x\| \rightarrow 0$  and  $\|T^{pn+p+1} x - T^{pn+1} x\| \rightarrow 0$

**Lemma 3.3 :** Let  $A_1, A_2, \dots, A_p$  be nonempty, closed subsets of a metric space and  $T: \bigcup_{i=1}^p A_i \rightarrow \bigcup_{i=1}^p A_i$  be a map

satisfying the conditions of the theorem 3.1 then If for some  $x \in A_j$  the sequence  $\{T^{pn} x\}$  in  $A_j$  contains a convergent subsequence  $\{T^{pn_j} x\}$ , converging to  $\xi \in A_j$ , then  $\xi$  is a best proximity point of  $T$  in  $A_j$ .

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Vol. 6, Issue 1, January 2017

**Proof:** we have  $d(T^{pn} x, T^{pn+1} x) \rightarrow \text{dist}(A_j, A_{j+1})$

$$\begin{aligned} \text{Now } \text{dist}(A_j, A_{j+1}) &\leq d(\xi, T\xi) = \lim_{n \rightarrow \infty} d(T^{pn_j} x, T\xi) \\ &\leq \lim_{n \rightarrow \infty} k d(T^{pn_j-1} x, \xi) + (1-k) \text{dist}(A_j, A_{j+1}) \\ &= \lim_{n \rightarrow \infty} k d(T^{pn_j-1} x, T^{pn_j} x) + (1-k) \text{dist}(A_j, A_{j+1}) \\ &\leq \lim_{n \rightarrow \infty} k [k d(T^{pn_j-2} x, T^{pn_j-1} x) + (1-k) \text{dist}(A_j, A_{j+1})] \\ &\quad + (1-k) \text{dist}(A_j, A_{j+1}) \\ &= \lim_{n \rightarrow \infty} k^2 d(T^{pn_j-2} x, T^{pn_j-1} x) + (1-k^2) \text{dist}(A_j, A_{j+1}) \\ &\leq \dots \lim_{n \rightarrow \infty} k^{pn_j} d(x, T y) + (1-k^{pn_j}) \text{dist}(A_j, A_{j+1}) \\ &\rightarrow \text{dist}(A_j, A_{j+1}) \end{aligned}$$

**Proof of theorem 3.1:** If  $\text{dist}(A_j, A_{j+1}) = 0$ , then by theorem 1.1,  $T$  has a unique fixed point and the theorem will be trivially true. So we assume that  $\text{dist}(A_j, A_{j+1}) > 0$ . Let  $x \in A_j$ , then by lemma 3.2  $\|T^{pn} x - T^{pn+1} x\| \rightarrow$

$\text{dist}(A_j, A_{j+1})$ . In view of lemma 3.3, to prove the theorem it is enough to prove that the sequence  $\{T^{pn} x\}$  in

$A_j$ , is Cauchy. Now in view of lemma 2.1 It will be sufficient to show that for given  $\epsilon > 0$  there exists  $N$  such that, for  $m > n \geq N$

$$\|T^{pm} x - T^{pn+1} x\| \leq \text{dist}(A_j, A_{j+1}) + \epsilon \quad \dots(1)$$

On contrary suppose (1) is not true then there exists  $\epsilon_0 > 0$  such that for every natural number  $k$ , there exists

$m_k > n_k \geq k$  such that

$$\|T^{pm_k} x - T^{pn_k+1} x\| \geq \text{dist}(A_j, A_{j+1}) + \epsilon_0$$

Here we assume that  $m_k$  is the smallest integer greater than  $n_k$  to satisfy the above inequality.

$$\text{Now } \text{dist}(A_j, A_{j+1}) + \epsilon_0 \leq \|T^{pm_k} x - T^{pn_k+1} x\| \leq \|T^{pm_k} x - T^{pm_k-p} x\| + \|T^{pm_k-p} x - T^{pn_k+1} x\|$$

Now  $\|T^{pm_k} x - T^{pm_k-p} x\| \rightarrow 0$  by remark 3.3,

this implies  $\|T^{pm_k} x - T^{pn_k+1} x\| \rightarrow \text{dist}(A_j, A_{j+1}) + \epsilon_0$

$$\text{Now } \|T^{pm_k} x - T^{pn_k+1} x\| \leq \|T^{pm_k} x - T^{pm_k+p} x\| + \|T^{pm_k+p} x - T^{pn_k+p+1} x\| + \|T^{pn_k+p+1} x - T^{pn_k+1} x\|$$

By remark 3.3  $\|T^{pm_k} x - T^{pm_k+p} x\| \rightarrow 0$  and  $\|T^{pn_k+p+1} x - T^{pn_k+1} x\| \rightarrow 0$

$$\lim_{k \rightarrow \infty} \|T^{pm_k} x - T^{pn_k+1} x\| \leq \lim_{k \rightarrow \infty} \|T^{pm_k+p} x - T^{pn_k+p+1} x\| \leq$$

$$\lim_{k \rightarrow \infty} k^p \|T^{pm_k} x - T^{pn_k+1} x\| + (1-k^p) \text{dist}(A_j, A_{j+1})$$

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$$\Rightarrow \text{dist}(A_j, A_{j+1}) + \epsilon_0 \leq k^p (\text{dist}(A_j, A_{j+1}) + \epsilon_0) + (1 - k^p) \text{dist}(A_j, A_{j+1}) + \epsilon_0$$

$$\Rightarrow \epsilon_0 \leq k^p \epsilon_0, \text{ which is a contradiction}$$

Hence the theorem.

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## BIOGRAPHY

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